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# Face-centered cubic magnetic photonic crystals: Some general electromagnetic properties followed from symmetry

Victor Dmitriev *

*Department of Electrical Engineering, University Federal of Para, P.O. Box 8619, Agencia UFPA, CEP 66075-900 Belem, Para, Brazil*

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**Abstract**

We discuss in this paper some general properties of magnetic photonic crystals with face-centered cubic lattice. Using the theory of magnetic groups, we consider the problem of changing symmetry of the crystals by a dc magnetic field and qualitative characteristics of their eigenwaves. The analysis is fulfilled using symmetry arguments alone.
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## 1. Introduction

Tunable and nonreciprocal optical and microwave photonic (electromagnetic) crystals have received much attention in the past decade. Tuning some parameters of photonic crystals and microcavities can be fulfilled by different means, and one of them is a dc magnetic field in case of magnetic materials. Using this method, one can for example, modify the size and position of bandgaps, open or close them, tune the frequency of microcavities. Magnetic photonic crystals also possess nonreciprocal properties. Possible applications of these crystals are modulators, switches, tunable phase shifters and filters, dispersion compensators, mode convertors, isolators and circulators, sensors.

Face-centered cubic (FCC) *electronic* magnetic crystals have been discussed in several papers. The symmetry properties of ferromagnetic metal FCC structures with three different orientation of magnetization, namely, parallel to [0 0 1], [1 1 1] and [1 1 0] have been described in ref. [1]. In particular, the basic domains of the Brillouin zone for these cases and the magnetic groups of the vector **k** for the points and the lines of symmetry in the Brillouin zone have been defined in this paper.

Many papers on magnetic *photonic* crystals were published recently (see for example, a review [2]). 3D magnetic crystals have been studied in refs. [3–6]. The influence of the magnetic anisotropy on the properties of these crystals taking into account the diagonal elements of the dielectric tensor has been analyzed in ref. [3]. The case of magnetization parallel to [0 0 1] and two directions of the electromagnetic wave propagation, namely, [0 0 1] and [0 1 0] have been studied. In the numerical calculations, two lower photonic bands have been considered. A new magneto-optic effect similar to the Faraday rotation but in the direction perpendicular to the biasing magnetic field has been theoretically predicted. This effect can be explained by multiple reflections of

* Tel.: +55 91 32017302.
*E-mail address:* victor@ufpa.br.

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the electromagnetic waves and by the magnetic symmetry of the crystal. Another new effect of opening up of multiple band gaps due to the applied dc magnetic field has been described in ref. [5].

Notice that due to multiple reflections of electromagnetic waves, a great enhancement of the Faraday and Kerr effects can be achieved in magnetic photonic crystals and this is important in optical region where the activity of ferrites is small. For example, the experimental work [6] with FCC optical photonic crystal made of silica spheres impregnated with a magnetic liquid has shown that inside the stop band around 600 nm, the Faraday rotation is enhanced by up to a factor of five.

Group theory helps us to understand the electromagnetic properties of photonic crystals. In particular, this theory has been used to classify the spatial symmetry of the nonmagnetic crystal eigenmodes. The group theory and the theory of representations have also been used to explain the effect of uncoupled modes due to mismatching of spatial symmetries of these modes and of the external exciting plane wave. Symmetry of eigenmodes defines selection rules in nonlinear processes. Group theory is also useful in numerical calculations. In the analysis of magnetic photonic crystals, one should use magnetic groups [7] taking into account the antiunitary nature of the Time reversal.

Because of the complexity of the magnetic photonic crystal problems, it is useful to analyze symmetry properties of these crystals. This gives an important qualitative information about them. This information can help to predict physical effects in the crystals, to define some features of their band diagrams and to investigate the properties of electromagnetic eigenwaves, to construct approximate solutions, etc. In numerical calculations, it is also useful to overcome the so-called "band indexing difficulties".

The main aim of this paper is to apply the theory of magnetic groups for investigations of 3D magnetic photonic crystals with FCC lattice. We shall discuss in this paper possible magnetic symmetries of these crystals, symmetry of the wave vectors, bidirectionality and lifting of eigenwave degeneracies by a dc magnetic field.

## 2. Time reversal operator

The time reversal operator $T$ which changes the sign of time $t$ (i.e. $Tt \to -t$) plays an important role in the description of electromagnetic properties of different physical media and devices, especially in those which are made of magnetic materials such as ferrites and magnetized plasma. One can meet in the literature different schemes of application of the Time reversal to

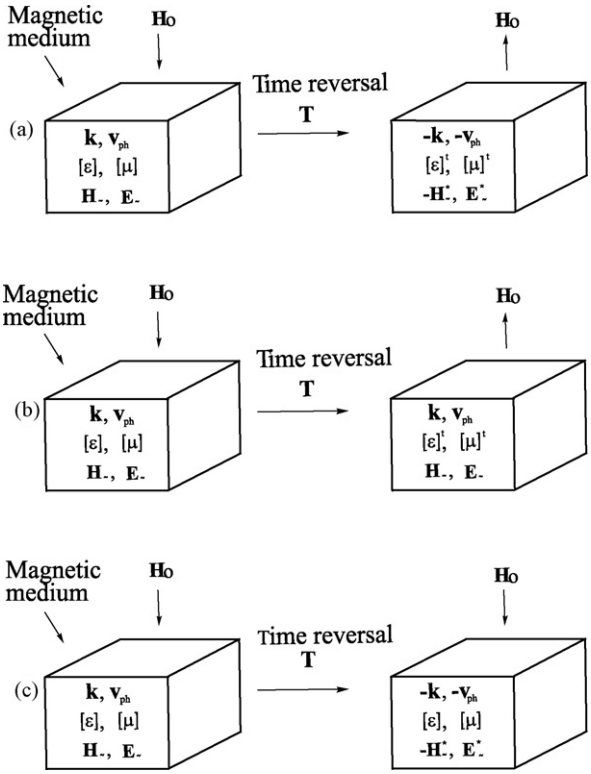


Fig. 1. The effect of the Time reversal operator on magnetic media and on electromagnetic wave: (a) the "full" Time reversal scheme, (b) and (c) are two "reduced" schemes, see Section 2.

electromagnetics. Three of them mentioned in ref. [8] are shown in Fig. 1. In the "full" time reversal scheme on Fig. 1a, the operator $T$ changes the signs of all the quantities which are odd in Time, i.e. of the external dc magnetic field $\mathbf{H}_0$, and consequently, transposes the material tensors $[\mu]$ and $[\epsilon]$, and also changes the signs of the ac magnetic field $\mathbf{H}_\sim$, of the wave vector $\mathbf{k}$ and of the phase velocity $\mathbf{v}_{\mathrm{ph}}$ [9]. This scheme is used for analysis of the time reversibility of nonlinear media [8].

In the two "reduced" schemes of Fig. 1b and c, the Time reversal operator "does not make the complete job". For example, in contrast to the scheme of Fig. 1a, the scheme of Fig. 1b preserves the signs of $\mathbf{k}$ and $\mathbf{v}_{\mathrm{ph}}$, i.e. preserves the direction of propagation of the electromagnetic wave but changes the sign of $\mathbf{H}_0$ [10]. In Fig. 1c, on the contrary, the operator $T$ does not affect the external dc magnetic field $\mathbf{H}_0$ and material parameters of the medium, but it changes the direction of propagation of the electromagnetic wave (see, for example ref. [11]). Both "reduced" schemes can be used for discussion of nonreciprocity. In this paper, we adopt the scheme of Fig. 1c.

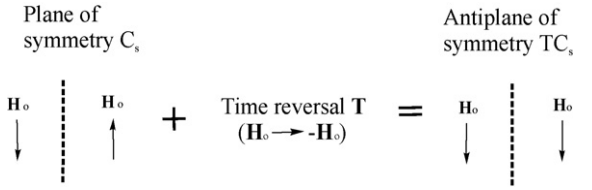


Fig. 2. The effect of the combined operator $TC_s$ on dc magnetic field $\mathbf{H}_0$.

It should be noted however that there is a subtlety in application of the operator $T$ in the scheme of Fig. 1c. In order to preserve the direction of the external dc magnetic field $\mathbf{H}_0$ inside the medium, in the combined operators (Time reversal + geometry transformation) we do change the sign of $\mathbf{H}_0$. For example, for antiplane of symmetry $TC_s$, it is demonstrated in Fig. 2. Thus, in our scheme, the effect of $T$ on $\mathbf{H}_0$ outside and inside of the medium is different.

Notice also that in the case of lossy media, a useful tool is the so-called restricted Time reversal operator [12].

## 3. Symmetry description of FCC magnetic lattices

The analyzed FCC structure shown in Fig. 3 consists of ferrite spheres immersed in a homogeneous dielectric.

Magnetization of the crystal by a dc magnetic field changes the symmetry of the crystal. We shall consider magnetization of the crystal by a uniform dc magnetic field $\mathbf{H}_0$.

The magnetic group of symmetry of the magnetic crystal depends on the orientation of $\mathbf{H}_0$ with respect to the symmetry axes and planes of the cube. For example, for $\mathbf{H}_0 \| x$, or $\mathbf{H}_0 \| y$ or $\mathbf{H}_0 \| z$ (Fig. 3), i.e. when $\mathbf{H}_0$ is parallel to one of the four-fold axes $C_4$, the resulting group is $D_{4h}(C_{4h})$ (in Schönflies notations [7]). If $\mathbf{H}_0$ parallel to one of the three-fold axes $C_3$, the magnetic group of symmetry is $D_{3d}(C_{3i})$. If $\mathbf{H}_0$ parallel to one of the two-fold axes $C''_2$, the magnetic group is $D_{2h}(C_{2h})$. If the magnetic field $\mathbf{H}_0$ parallel to one of the planes of symmetry $\sigma_v$, $\sigma_d$ or $\sigma_h$ but it does not coincide with any axis lying in this plane, the crystal has the symmetry $C_{2h}(C_i)$. For any other orientation of $\mathbf{H}_0$, the magnetic group of symmetry is $\mathbf{C}_i$ consisting of 2 elements with no Time reversal. This information is summarized in Table 1. In what follows, we shall restrict ourselves by discussion of the magnetic crystals for magnetization $\mathbf{H}_0 \| z$ (Fig. 4b). This structure posesses the symmetry $D_{4h}(C_{4h})$. The elements of the magnetic group $D_{4h}(C_{4h})$ are given in Fig. 4b. For comparison, the elements of the nonmagnetic symmetry $O_h$ are presented in Fig. 4a. Comparing Fig. 4a and b, we see that magnetization of the crystal by the magnetic field leads to a significant reduction of symmetry.

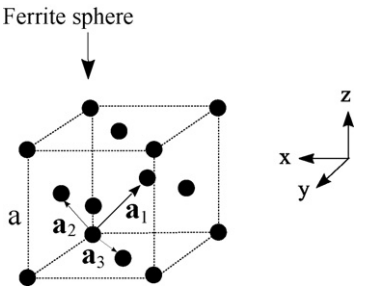


Fig. 3. FCC lattice of ferrite spheres.

Table 1
Symmetry of FCC magnetic crystals for different orientations of dc magnetic field $\mathbf{H}_0$

| Orientation of $\mathbf{H}_0$ | Magnetic group |
|---|---|
| $\mathbf{H}_0 \| z$ | $D_{4h}(C_{4h})$ |
| $\mathbf{H}_0 \| C_3$ | $D_{3d}(C_{3i})$ |
| $\mathbf{H}_0 \| C''_2$ | $D_{2h}(C_{2h})$ |
| $\mathbf{H}_0 \|$ to any plane | $C_{2h}(C_i)$ |
| Arbitrary | $\mathbf{C}_i$ |

With the uniform magnetization, the Bravais lattice and the Brillouin zone of the magnetized FCC crystal do not differ from those of the nonmagnetic crystal, and this

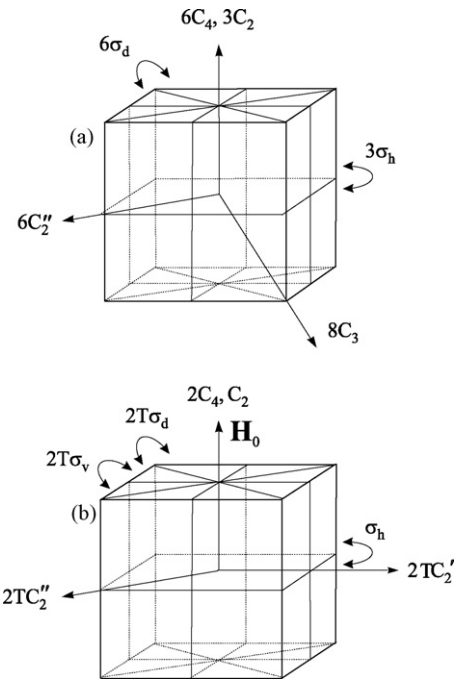


Fig. 4. Symmetry operations (a) of the group $O_h$ [15], (b) of the group $D_{4h}(C_{4h})$.

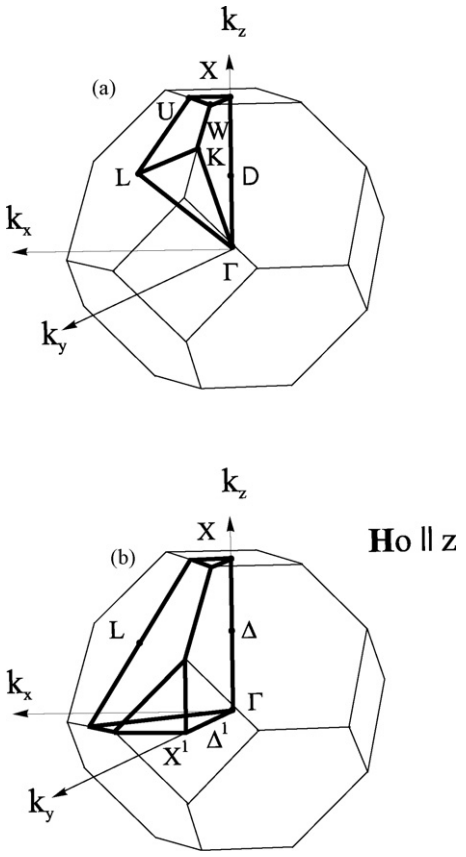


Fig. 5. The basic domains for FCC crystals: (a) nonmagnetic case, the group $O_h$, (b) magnetic case with magnetization along the $z$-axis [0 0 1], the magnetic group $D_{4h}(C_{4h})$ [1].

is true for any direction of dc magnetic field $\mathbf{H}_0$. However, the size and the form of the basic domain depend on the orientation of $\mathbf{H}_0$. For comparison, we show in Fig. 5 the basic domains for the nonmagnetic case and for the magnetic crystal with $\mathbf{H}_0 \| z$.

The Brillouin zone of the nonmagnetized FCC crystal is the truncated octahedron. The group of symmetry of the nonmagnetic structure is $O_h + TO_h$ where $T$ the time reversal operator. The group of symmetry of the point Γ is $O_h$, of the point $X$ is $D_{4h}$ and of the point $L$ is $D_{3d}$.

In the case of $\mathbf{H}_0 \| z$, the magnetic group of symmetry of the crystals is $D_{4h}(C_{4h})$, the points Γ and $X$ have also the symmetry $D_{4h}(C_{4h})$, and the point $L$ has the symmetry $C_{2h}(C_i)$.The little magnetic groups and their elements for some points and lines of symmetry for this case are given in Table 2.

For lower symmetries, the basic domain of the Brillouin zone increases. For example, the basic domain for the group $\mathbf{C}_i$ is a half of the Brillouin zone.

## 4. Eigenwave equation

We shall consider Maxwell's equations in the frequency domain. Symmetry properties of the crystals can be discussed in terms of **D** and **B** vectors or equivalently, in terms of **E** and **H** vectors. We write the vector **B** as a plane wave

$$\mathbf{B} = \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r})}\mathbf{u}_{\mathbf{k}n}(\mathbf{r}), \tag{1}$$

where $\mathbf{u}_{\mathbf{k}n}(\mathbf{r})$ is a periodic function with the period of the crystal, $n$ is a band index.

The eigenvalue equation which relates the wave vector **k** and the frequency $\omega$ can be easily obtained in the following form:

$$(i\boldsymbol{k} + \nabla) \times \{\epsilon^{-1}(\boldsymbol{r})(i\boldsymbol{k} + \nabla) \times [\bar{\boldsymbol{\mu}}^{-1}(\boldsymbol{r})\boldsymbol{u}_{\boldsymbol{k}n}(\boldsymbol{r})]\} = \frac{\omega_n^2}{c^2}\boldsymbol{u}_{\boldsymbol{k}n}(\boldsymbol{r}), \tag{2}$$

or

$$\mathcal{L}_u \mathbf{u}_{\mathbf{k}n}(\mathbf{r}) = \frac{\omega_n^2}{c^2}\mathbf{u}_{\mathbf{k}n}(\mathbf{r}), \tag{3}$$

where the tensor of the second rank $\bar{\mu}(\mathbf{r})$ is the permeability and the scalar $\epsilon(\mathbf{r})$ is the permittivity.

Analogously, writing the vector **D** as

$$\boldsymbol{D} = \mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{r})}\boldsymbol{v}_{\boldsymbol{k}n}(\boldsymbol{r}), \tag{4}$$

we can obtain the following wave equation:

$$(i\boldsymbol{k} + \nabla) \times \{\bar{\boldsymbol{\mu}}^{-1}(\boldsymbol{r})(i\boldsymbol{k} + \nabla) \times [\epsilon^{-1}(\boldsymbol{r})\boldsymbol{v}_{\boldsymbol{k}n}(\boldsymbol{r})]\} = \frac{\omega_n^2}{c^2}\boldsymbol{v}_{\boldsymbol{k}n}(\boldsymbol{r}), \tag{5}$$

Table 2
Little magnetic groups and their elements for some points and lines of symmetry for FCC lattice magnetized along the $z$-axis, the group of the lattice is $D_{4h}(C_{4h})$

| Symmetry symbol | Representative wave vector $\boldsymbol{k}$ | Magnetic little group | Order of the group | Elements of the group |
|---|---|---|---|---|
| Γ | $(\pi/a)(0, 0, 0)$ | $D_{4h}(C_{4h})$ | 16 | $e, 2C_4, C_2, i, 2S_4, \sigma_h,$ |
| $X$ | $(\pi/a)(0, 0, 1)$ | | | $2TC'_2, 2TC''_2, 2T\sigma_v, 2T\sigma_d$ |
| $X'$ | $(\pi/a)(0, 1, 0)$ | $D_{2h}(C_{2h})$ | 8 | $e, C_2, i, \sigma_h, 2TC'_2, 2T\sigma_v$ |
| Δ | $(\pi/a)(0, 0, \alpha)$ | $D_4(C_4)$ | 8 | $e, 2C_4, C_2, 2TC'_2, 2TC''_2$ |
| Δ′ | $(\pi/a)(0, \alpha, 0)$ | $C_{2v}(C_s)$ | 4 | $e, \sigma_h, TC'_2, T\sigma_v$ |

or

$$\mathcal{L}_v \boldsymbol{v}_{\boldsymbol{k}n}(\boldsymbol{r}) = \frac{\omega_n^2}{c^2}\boldsymbol{v}_{\boldsymbol{k}n}(\boldsymbol{r}). \tag{6}$$

It can be shown that the operators $\mathcal{L}_u$ and $\mathcal{L}_v$ are invariant with respect to the symmetry operators corresponding to the elements of the little magnetic groups. Eqs. (2) and (5) can be solved numerically to obtain the band diagrams $\omega_n(\mathbf{k})$ and then to calculate the electromagnetic fields.

## 5. Reciprocity and bidirectionality of crystal

For nonmagnetic crystals, the Time reversal symmetry allows one to reduce the burden of numerical calculations by half, because the dispersion characteristics are symmetrical with respect to changing the sign of $\mathbf{k}$, i.e. $\omega(\mathbf{k}) = \omega(-\mathbf{k})$.

The groups of symmetry of magnetic crystals do not contain the Time reversal operator in "pure" form. It can be present in these groups only in combinations with some geometric transformations.

For any orientation of the uniform dc magnetic field $\mathbf{H}_0$, the resulting magnetic group of symmetry of the FCC crystals always contains the inversion $i$, because both the group of the vector $\mathbf{H}_0$ and the group of nonmagnetic crystal $O_h$ contain this element of symmetry. The operator $i$ changes the sign of $\mathbf{k}$. Therefore, the dispersion characteristics will be symmetrical with respect to the changing of the sign of wave vector $\mathbf{k}$, i.e. $\omega_n(\mathbf{k}) = \omega_m(-\mathbf{k})$. A crystal with this property can be called bidirectional [13]. The symmetry of the band diagrams with respect to the sign of the vector $\mathbf{k}$ does not mean that the crystal is reciprocal. We use in the equality $\omega_n(\mathbf{k}) = \omega_m(-\mathbf{k})$ different subindexes $n$ and $m$, because, in general, the structure of the electromagnetic field of the eigenwaves corresponding to $\mathbf{k}$ and $-\mathbf{k}$ is different. Nonreciprocity of the crystal can manifest itself also in different losses of the waves with $\mathbf{k}$ and $-\mathbf{k}$.

## 6. Compatibility relations and lifting of eigenwave degeneracy by dc magnetic field

For nonmagnetic and also for magnetic crystals, compatibility relations show the mutual relations between the irreducible representations for adjacent $\mathbf{k}$ vectors. For construction of the compatibility tables in the case of magnetic symmetry $G(H)$, we can use only the unitary subgroups $H$ following the procedure described in many books on group theory.

In case of magnetic crystals, we can consider two other types of compatibility relations. First of all, one can compare the symmetries of a wave vector $\mathbf{k}$ in nonmagnetic and magnetic states of the crystal considering magnetization as a perturbation of the crystal. The corresponding compatibility tables will show in particular, the possible lifting of eigenwave degeneracy by dc magnetic field (see below).

Still another type of compatibility relations can be considered for magnetic crystals, namely, the relations which are defined by deviation of the direction of dc magnetic field from a given one. As a result of this deviation, the groups of symmetry of the crystal and of the vector $\mathbf{k}$ are changed and this leads to a qualitative change of the dispersion diagrams. Thus, one can consider the mutual relations between the irreducible representations for different orientation of dc magnetic field.

Let us define as an example the compatibility relations for nonmagnetic and magnetic states of our crystals. We shall consider two directions in the magnetic crystal: (a) along the line $\Delta$ and (b) along the line $\Delta'$ (see Figs. 5b and 6). The group of symmetry of $\mathbf{k}$ on the line $\Delta$ in nonmagnetic case is $C_{4v} + TC_{4v}$ and in magnetic

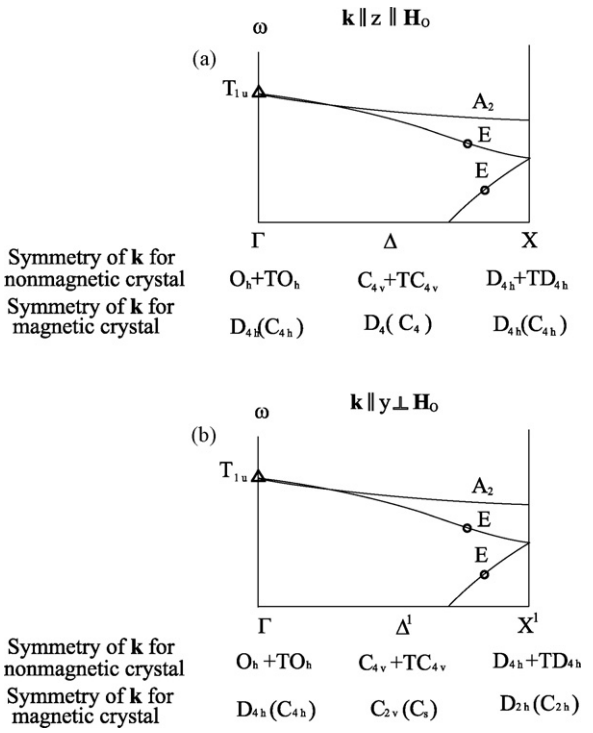


Fig. 6. The lowest bands of the dispersion diagrams for nonmagnetic FCC crystal with symmetry $O_h + TO_h$, adapted from [14]. The points marked by circles correspond to the doubly degenerate representation $E$ and the points marked by triangles correspond to the triply degenerate representations $T_{1g}$ and $T_{1u}$. Two lines below the diagrams show symmetries of the wave vector for nonmagnetic and magnetic crystal: (a) along the line $\Delta$ and (b) along the line $\Delta'$.

Table 3
Compatibility relations for nonmagnetic and magnetic FCC lattice, the line $\Delta$

| | | | | | |
|---|---|---|---|---|---|
| $C_{4v}$ | $A_1$ | $A_2$ | $B_1$ | $B_2$ | $E$ |
| $C_s$ | $A'$ | $A''$ | $A'$ | $A''$ | $A' + A''$ |

Table 4
Compatibility relations for nonmagnetic and magnetic FCC lattice, the line $\Delta'$

| | | | | | |
|---|---|---|---|---|---|
| $C_{4v}$ | $A_1$ | $A_2$ | $B_1$ | $B_2$ | $E$ |
| $C_4$ | $A$ | $A$ | $B$ | $B$ | ${}^1E{}^{+2}E$ |

Table 5
Compatibility relations for nonmagnetic and magnetic FCC lattice, the point $\Gamma$

| | | | | | |
|---|---|---|---|---|---|
| $O_h$ | $A_{1g}$ | $A_{2g}$ | $E_g$ | $T_{1g}$ | $T_{2g}$ |
| $C_{4h}$ | $A_g$ | $B_g$ | $A_g + B_g$ | $A_g + {}^1E_g + {}^2E_g$ | $B_g + {}^1E_g + {}^2E_g$ |
| $A_{1u}$ | $A_{2u}$ | $E_u$ | $T_{1u}$ | $T_{2u}$ | |
| $A_u$ | $B_u$ | $A_u + B_u$ | $A_u + {}^1E_u + {}^2E_u$ | $B_u + {}^1E_u + {}^2E_u$ | |

case is $C_{2v}(C_s)$, and on the line $\Delta'$, these groups are $C_{4v} + TC_{4v}$ and $D_4(C_4)$, respectively. The irreducible representations of these point groups can be found in ref. [7].

Using the standard procedure of compatibility table construction, we calculate Tables 3–5 .

We can see from these tables, that doubly degenerate representations $E$ (the corresponding characteristics are denoted by circles in Fig. 6) for the lines $\Delta$ and $\Delta'$ split into one-dimensional representations. Thus, the applied magnetic field will split the dispersion characteristics into two separate curves. Analogously, the triply degenerate modes $T_{1u}$ in the points $\Gamma$ marked by triangles in Fig. 6a and b will split by the magnetic field into three different modes.

## 7. Conclusions

The main results of our paper are as follows. We have defined all the possible magnetic groups of FCC magnetic photonic crystals magnetized by a uniform dc magnetic field. We have introduced a new type of compatibility relations for magnetic crystals defined by different orientation of dc magnetic field. We have discussed the problems of nonreciprocity and bidirectionality in the crystals and lifting of eigenvalue degeneracy by applied dc magnetic field.

With small modifications (it concerns the wave Eqs. (2) and (5) where one should make the following substitutions: $\bar{\mu} \to \mu$ and $\epsilon \to \bar{\epsilon}$), the symmetry analysis of this paper can be applied also to the FCC photonic crystals with magnetized semiconductor elements.

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